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WEIGHTED WEAK TYPE INTEGRAL INEQUALITIES FOR THE HARDY–LITTLEWOOD MAXIMAL OPERATOR

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ABSTRACT

In this paper we characterize the pairs of weights (u, w) for which the Hardy–Littlewood maximal operator M satisfies a weak type integral inequality of the form

$$\int_{\{x\in\mathbf{R}^n:Mf(x)>\lambda\}} u\,dx \leqq \frac{C}{\phi(\lambda)}\int_{\mathbf{R}^n}\phi(|f|)w\,dx,$$

with C independent of f and $\lambda>0$, where ϕ is an N-function. Moreover, for a given weight w, a necessary and sufficient condition is found for the existence of a positive weight u such that M satisfies an integral inequality as above. Lastly, in the case $u=w$, we notice that the conclusion of the extrapolation theorem given by J. L. Rubio de Francia, which appeared in Am. J. Math. **106** (1984), can be strengthened to Orlicz spaces.

1. Introduction

Let M be the Hardy–Littlewood maximal operator defined by

$$(1.1)\qquad Mf(x)=\sup_{x\in Q}\frac{1}{|Q|}\int_Q |f|\,dx \qquad (f\in L^1_{\mathrm{loc}}(\mathbf{R}^n)),$$

where the supremum is taken over all cubes Q containing x and $|Q|$ is the Lebesgue measure of Q. (Cube will always mean a compact cubic interval with nonempty interior.)

Our main aim is to study weak type integral inequalities with weights for M. More exactly, we extend the result of Theorem 1 of B. Muckenhoupt in [8], for

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$L_p(\mathbf{R})$, to the context of Orlicz spaces. Muckenhoupt's result, extended to $\mathbf{R}^n$ (see Theorem IV–1.12 in [3]), asserts that, *given* $p > 1$ *and* u, w *two weights on* $\mathbf{R}^n$, M *is of weak type* (p, p) *with respect to the measures udx and wdx, that is, there exists a positive constant C such that for every* $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ *and every* $\lambda > 0$

$$\int_{u\{x\in\mathbf{R}^n: Mf(x)>\lambda\}} udx \leqq C\lambda^{-p}\int_{\mathbf{R}^n}|f|^p w dx$$

if and only if (u, w) *satisfies* A_p, *that is, there is a constant K such that for every cube Q we have*

$$(1.2)\qquad \left(\frac{1}{|Q|}\int_Q udx\right)\left(\frac{1}{|Q|}\int_Q w^{-1/(p-1)}dx\right)^{p-1} \leqq K.$$

By *a weight* on $\mathbf{R}^n$ we mean a Lebesgue-measurable function with values in $[0, \infty)$. Sometimes, we shall write $u(E)$ for $\int_E udx$.

In this paper we characterize the pairs of weights (u, w) on $\mathbf{R}^n$ which satisfy an integral inequality of the form

$$(1.3)\qquad u\{x\in\mathbf{R}^n : Mf(x)>\lambda\} \leqq \frac{C}{\phi(\lambda)}\int_{\mathbf{R}^n}\phi(|f|)wdx,$$

where ϕ is an N-function. The characterizing condition is the natural two-weight analogue of the A_ϕ-condition introduced by R. Kerman and A. Torchinsky [6] to characterize the one-weight strong type inequality for Orlicz spaces. When (1.3) holds, for every $f\in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ and every $\lambda > 0$, we shall say that M *is of weak type* (ϕ, ϕ) *with respect to* (u, w).

We also characterize the weights w for which there is a positive weight u such that M is of weak type (ϕ, ϕ) with respect to (u, w) and we shall finish, in the case $u = w$, with an extrapolation result in the theory of weights.

Now, we shall present the basic definitions and results concerning N-functions and Orlicz spaces which will be used in this paper.

An N-function is a continuous and convex function $\phi : [0, \infty) \to \mathbf{R}$ *such that* $\phi(s) > 0$, $s > 0$, $s^{-1}\phi(s) \to 0$ *for* $s \to 0$ *and* $s^{-1}\phi(s) \to \infty$ *for* $s \to \infty$.

An N-function ϕ has the representation $\phi(s) = \int_0^s \varphi$ where $\varphi : [0, \infty) \to \mathbf{R}$ is continuous from the right, non-decreasing such that $\varphi(s) > 0$, $s > 0$, $\varphi(0) = 0$ and $\varphi(s) \to \infty$ for $s \to \infty$. More precisely φ is the right derivate of ϕ and will be called the *density function of* ϕ.

Associated to φ we have the function $\rho : [0, \infty) \to \mathbf{R}$ defined by $\rho(t) = \sup\{s : \varphi(s) \leqq t\}$ which has the same aforementioned properties of φ. We will call ρ *the generalized inverse of* φ.

We have $\varphi(\rho(t)) \geqq t$, $t \geqq 0$, and $\varphi(\rho(t)-\varepsilon) \leqq t$ for every positive reals t and ε such that $\rho(t)-\varepsilon \geqq 0$.

The N-function ψ defined by $\psi(t)=\int_0^t \rho$ is called *the complementary N-function of* ϕ. Thus, if $\phi(s)=p^{-1}s^p$, $p>1$, then $\psi(t)=q^{-1}t^q$ where $pq=p+q$.

Young's inequality asserts that $st \leqq \phi(s)+\psi(t)$ for $s, t \geqq 0$, equality holding if and only if $\varphi(s-) \leqq t \leqq \varphi(s)$ or else $\rho(t-) \leqq s \leqq \rho(t)$.

An N-function ϕ is said to satisfy the Δ_2-*condition* in $[0,\infty)$ (or merely the Δ_2-condition) if $\sup_{s>0} \phi(2s)/\phi(s) < \infty$. If φ is the density function of ϕ, then ϕ satisfies Δ_2 if and only if there exists a constant $\alpha>1$ such that $s\varphi(s)<\alpha\phi(s)$, $s>0$. The Δ_2-condition for ϕ does not transfer necessarily to the complementary N-function. The latter satisfies the Δ_2-condition if and only if there exists a constant $\beta>1$ such that $\beta\phi(s)<s\varphi(s)$, $s>0$.

If $(X, \mathcal{M}, \mu)$ is a σ-finite measure space we denote by $\mathfrak{M}$ the space of $\mathcal{M}$-measurable and μ-a.e. finite functions from X to $\mathbf{R}$ (or to $\mathbf{C}$). If ϕ is an N-function *the Orlicz spaces* $L_\phi(\mu) \equiv L_\phi(X, \mathcal{M}, \mu)$ and $L_\phi^*(\mu) \equiv L_\phi^*(X, \mathcal{M}, \mu)$ are defined by

$$L_\phi(\mu)=\left\{f\in\mathfrak{M} : \int_X \phi(|f|)\,d\mu<\infty\right\}$$

and

$$L_\phi^*(\mu)=\{f\in\mathcal{M} : fg\in L_1(\mu) \text{ for all } g\in L_\psi\},$$

where ψ is the complementary N-function of ϕ.

When $\mu=wdx$ for a weight w on $\mathbf{R}^n$ we write $L_\phi(w)$ for $L_\phi(\mu)$.

We have $L_\phi(\mu) \subset L_\phi^*(\mu)$ and if ϕ satisfies Δ_2, then $L_\phi(\mu)=L_\phi^*(\mu)$.

The Orlicz space $L_\phi^*(\mu)$ is a Banach space with the norms $\|f\|_\phi = \sup\{\int_X |fg|\,d\mu : g\in S_\psi\}$, where $S_\psi=\{g\in L_\psi : \int_X \psi(|g|)d\mu \leqq 1\}$, and $\|f\|_{(\phi)} = \inf\{\lambda>0 : \int_X \phi(\lambda^{-1}|f|)d\mu \leqq 1\}$, which are called *the Orlicz norm* and *the Luxemburg norm*, respectively. Both norms are equivalent, actually $\|f\|_{(\phi)} \leqq \|f\|_\phi \leqq 2\|f\|_{(\phi)}$.

Holder's inequality asserts that for every $f\in L_\phi^*(\mu)$ and every $g\in L_\psi^*(\mu)$ we have $\|fg\|_1 \leqq \|f\|_{(\phi)}\|g\|_\psi$, where ϕ and ψ are complementary N-functions.

If $\phi(s)=s^p$ with $p>1$, then, $L_\phi^*(\mu)=L_\phi(\mu)=L_p(\mu)$, $\|f\|_{(\phi)}=\|f\|_p$ and $\|g\|_\psi=\|g\|_q$, where $pq=p+q$.

The proofs of above-mentioned results can be found in [7] or in IV–13 of [9].

2. The weak type and the A_ϕ-condition

DEFINITION 2.1. Let φ be the density function of the N-function ϕ, ρ the generalized-inverse of φ and let u and w be weights on X. We shall say that the pair (u, w) satisfies the A_ϕ-condition, or that it belongs to the A_ϕ-class, if there exists a positive constant C such that for every cube Q and every positive real ε

$$\left(\frac{1}{|Q|}\int_Q \varepsilon u dx\right)\varphi\left(\frac{1}{|Q|}\int_Q \rho(1/\varepsilon w)dx\right) \leqq C. \tag{2.2}$$

If for some cube Q, $\int_Q \rho(1/\varepsilon w) = \infty$ we assume that (2.2) holds if $\int_Q u dx = 0$; therefore, if u is not identically null (2.2) shows that $w(x) > 0$ for almost every $x \in \mathbf{R}^n$. In the same way, if for some cube Q, $\int_Q u dx = \infty$ we assume that (2.2) holds if $\int_Q \rho(1/\varepsilon w) = 0$ for every $\varepsilon > 0$, which implies that if (u, w) satisfies A_ϕ then u is locally integrable.

In the case $\phi(s) = p^{-1}s^p$, $p > 1$, (2.1) gives the classical A_p-condition since in this situation ε does not take part in (2.2) (ρ is multiplicative and moreover it is the inverse of φ) and thus (2.2) reduces to (1.2).

If ϕ satisfies the Δ_2-condition, then, every pair (u, w) of the A_1-class is in the A_ϕ-class. (We recall that (u, w) belongs to A_1-class if there is a constant K such that $Mu \leqq Kw$ a.e.)

In fact, if $(u, w) \in A_1$ there is a constant K such that for every cube Q

$$|Q|^{-1}\int_Q u dx \leqq K \operatorname*{ess\,inf}_{x \in Q} w(x).$$

Let Q be such that $u(Q) > 0$ and let $\beta = \operatorname{ess\,inf}\{w(x) : x \in Q\}$. We have

$$\left(|Q|^{-1}\int_Q \varepsilon u dx\right)\varphi\left(|Q|^{-1}\int_Q \rho(1/\varepsilon w)dx\right) \leqq \varepsilon\beta K\, \varphi(\rho(1/\varepsilon\beta)).$$

On the other hand, since ϕ satisfies Δ_2, there exists $a > 0$ such that $\phi(2s) \leqq a\phi(s)$ and $s\varphi(s) \leqq a\phi(s)$ for every $s \geqq 0$; therefore, there exists $\alpha > 0$ such that $\varphi(2s) \leqq \alpha\varphi(s)$, $s \geqq 0$, hence for every $t > 0$ and $\varepsilon > 0$ such that $\rho(t) \leqq 2(\rho(t) - \varepsilon)$ we have

$$\varphi(\rho(t)) \leqq \varphi(2(\rho(t) - \varepsilon)) \leqq \alpha\varphi(\rho(t) - \varepsilon) \leqq \alpha t.$$

Hence, it follows that (2.2) holds with $C = \alpha K$.

When (w, w) belongs to the A_ϕ-class we shall say, merely, that w satisfies A_ϕ (this is the terminology introduced in [6]).

In what follows we shall always assume that ϕ, *together with its complementary N-function, satisfy the Δ_2-condition*. Examples of N-functions in this situation are (among others): $\phi_1(s)=s^p$, $p>1$; $\phi_2(s)=s^p(1+\log(1+s))$, $p>1$; $\phi_3(s)=s^p(1+\log^+ s)$, $p>1$; $\phi_4(s)=\int_0^s \rho$ where $\rho:[0,\infty)\to[0,\infty)$ is defined by $\rho(0)=0$, $\rho(t)=2^{-n}$ if $t\in[2^{-n},2^{-n+1})$ and $\rho(t)=2^{n-1}$ if $t\in[2^{n-1},2^n)$, n a positive integer.

The condition A_ϕ characterizes the pairs of weights (u,w) for which the Hardy–Littlewood maximal operator M is of weak type (ϕ,ϕ) with respect to the measures $w\,dx$ and $u\,dx$; more precisely:

THEOREM 2.3. *Let u and w be weights on $\mathbf{R}^n$. The following conditions are equivalent:*

(a) *There exists C such that for every $f\in L^1_{\text{loc}}(\mathbf{R}^n)$ and every $\lambda>0$*

$$u\{x\in\mathbf{R}^n : Mf(x)>\lambda\}\leqq\frac{C}{\phi(\lambda)}\int_{\mathbf{R}^n}\phi(|f|)w\,dx.$$

(b) *There exists C such that for every $f\in L^1_{\text{loc}}(\mathbf{R}^n)$ and every cube Q*

$$\phi\left(\frac{1}{|Q|}\int_Q|f|\,dx\right)u(Q)\leqq C\int_Q\phi(|f|)w\,dx. \tag{2.4}$$

(c) *The pair (u,w) satisfies A_ϕ.*

NOTE. We use the convention that C denotes an absolute positive constant which may change from line to line.

PROOF

PROOF OF (a)$\Rightarrow$(c). It suffices to prove that for every cube Q we have

$$\left(|Q|^{-1}\int_Q u\,dx\right)\varphi\left(|Q|^{-1}\int_Q\rho(1/w)\,dx\right)\leqq C, \tag{2.5}$$

with C depending only on ϕ and the constant of condition (a).

Since we assume that ϕ and its complementary N-function ψ satisfy the Δ_2-condition, there exists a constant $\alpha>1$ such that $s\varphi(s)\leqq\alpha\phi(s)$ and $s\rho(s)\leqq\alpha\psi(s)$ for every $s\geqq0$.

If $u(x)=0$ a.e., then (2.5) is obvious and otherwise (a) implies that $w(x)>0$ a.e. Let Q be a cube such that $u(Q)>0$; then, the function $w^{-1}\chi_Q$ belongs to $L_\psi(w)$ since otherwise we would have that there is $g\in L_\phi(w)$ such that $gw^{-1}\chi_Q\notin L_1(w)$ and consequently $Mg(x)=\infty$ for every $x\in Q$, which implies that

$$u(Q)\leqq\lim_{\lambda\to\infty}\frac{C}{\phi(\lambda)}\int_Q\phi(|g|)w\,dx=0.$$

Therefore $0<\int_Q \rho(1/w)dx \leqq \alpha \int_{\mathbf{R}^n} \psi(w^{-1}\chi_Q)wdx<\infty$ and hence $\rho(w^{-1}\chi_Q)=\rho(1/w)\chi_Q$ belongs to $L^1_{\mathrm{loc}}(\mathbf{R}^n)$. (Moreover $\rho(w^{-1}\chi_Q)\in L_\phi(w)$, since $\phi(\rho(s))=s\rho(s)-\psi(s)\leqq(\alpha-1)\psi(s)$.) It follows from (a), taking $f=\rho(w^{-1}\chi_Q)$ and $\lambda=|Q|^{-1}\int_Q\rho(1/w)dx$, that

$$\phi\left(|Q|^{-1}\int_Q \rho(1/w)dx\right)u(Q)\leqq C\int_{\mathbf{R}^n}\phi(\rho(w^{-1}\chi_Q))wdx\leqq C\int_Q \rho(1/w)dx$$

and thus we get (2.5) with constant αC.

Proof of (c)$\Rightarrow$(b). Let Q be a cube such that $u(Q)>0, f\in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ and let $\varepsilon>0$. The A_ϕ-condition implies that $(\varepsilon w)^{-1}\chi_Q\in L_\psi(\varepsilon w)$ and then, using Hölder's inequality, we have

$$\int_Q |f|dx\leqq 2\,\| f\chi_Q\|_{(\phi),\varepsilon w}\cdot\|(\varepsilon w)^{-1}\chi_Q\|_{(\psi),\varepsilon w}, \tag{2.6}$$

where the norms used are those of Luxemburg with respect to the measure $\varepsilon w\,dx$.

On the other hand, for every $\lambda>0$ we get

$$\int_{\mathbf{R}^n}\psi((\lambda\varepsilon w)^{-1}\chi_Q)\varepsilon w\,dx\leqq\int_Q \lambda^{-1}\rho((\lambda\varepsilon w)^{-1})dx\leqq\lambda^{-1}|Q|\rho(CQ(\lambda\varepsilon u(Q))^{-1}),$$

where C is a constant in the A_ϕ-condition for (u,w). Therefore, for $\lambda=C|Q|\phi^{-1}(1/\varepsilon u(Q))$ and taking into account that $s\leqq\phi^{-1}(s)\psi^{-1}(s)$, $s\geqq 0$, we have

$$\begin{aligned}\int_{\mathbf{R}^n}\psi((\lambda\varepsilon w)^{-1}\chi_Q)\varepsilon w\,dx&\leqq\frac{1}{C\phi^{-1}(1/\varepsilon u(Q))}\,\rho\left(\frac{1}{\varepsilon u(Q)\phi^{-1}(1/\varepsilon u(Q))}\right)\\&\leqq\alpha C^{-1}\varepsilon u(Q)\psi\left(\frac{1/\varepsilon u(Q)}{\phi^{-1}(1/\varepsilon u(Q))}\right)\\&\leqq\alpha C^{-1},\end{aligned}$$

where $\alpha>1$ is such that $s\rho(s)\leqq\alpha\psi(s)$, $s\geqq 0$. We may assume that $C\geqq\alpha$ and therefore

$$\|(\varepsilon w)^{-1}\chi_Q\|_{(\psi),\varepsilon w}\leqq C|Q|\phi^{-1}(1/\varepsilon u(Q)).$$

Now, it follows from (2.6) that

$$|Q|^{-1}\int_Q|f|dx\leqq 2C\phi^{-1}(1/\varepsilon u(Q))\,\|f\chi_Q\|_{(\phi),\varepsilon w}.$$

Then, taking $\varepsilon = \varepsilon(f, Q) = (\int_Q \phi(|f|)w dx)^{-1}$ we have $\| f\chi_Q \|_{(\phi),\varepsilon w} = 1$ and, since C does not depend on f, Q and ε, we have

$$\frac{1}{|Q|} \int_Q |f| dx \leqq 2C\phi^{-1}\left(\frac{1}{u(Q)} \int_Q \phi(|f|)w dx\right),$$

whence (2.4) follows, taking into account that ϕ is increasing and satisfies Δ_2.

Proof of (b)$\Rightarrow$(a). The proof of this implication is easy using an appropriate covering theorem. We shall use a Besicovitch type covering theorem (see Theorem 1.1 in [4]).

Let N be a positive integer and M_N the truncated maximal operator defined by

$$M_N f(x) = \sup_{x \in Q;\, \delta(Q) \leqq N} \frac{1}{|Q|} \int_Q |f| dx,$$

where $\delta(Q)$ is the diameter of Q.

For $\lambda > 0$ and $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ let $A_{\lambda,N} = \{x \in \mathbf{R}^n : M_N f(x) > \lambda\}$. For every $x \in A_{\lambda,N}$ there is a cube $Q(x)$, centered at x, such that $|Q(x)|^{-1} \int_{Q(x)} |f| dx > a_n \lambda$, where the constant $a_n > 0$ only depends on the dimension n. Since $\sup\{\delta(Q(x)), x \in A_{\lambda,N}\} < \infty$, we can choose from $\{Q(x), x \in A_{\lambda,N}\}$ a sequence $\{Q_k\}$ (possibly finite) such that

$$A_\lambda \subset \bigcup_k Q_k, \qquad \sum_k \chi_{Q_k} \leqq C_n \chi_{\cup_k Q_k},$$

where C_n only depends on n.

Thus, (b) shows that

$$\begin{aligned} u\{x \in \mathbf{R}^n : Mf(x) > \lambda\} &= \lim_{N \to \infty} u(A_{\lambda,N}) \\ &\leqq \frac{C}{\phi(\lambda)} \sum_k \phi\left(\frac{1}{|Q_k|} \int_{Q_k} |f| dx\right) u(Q_k) \\ &\leqq \frac{C}{\phi(\lambda)} \int_{\mathbf{R}^n} \phi(|f|) w dx, \end{aligned}$$

where C depends only on n, ϕ and the constant in condition (b), which proves (a). Thus, the proof is complete.

Remarks. The equivalence (b)$\Leftrightarrow$(c) gives some interesting consequences: (1) If there exists a positive weight u, such that (u, w) satisfies A_ϕ, then,

$L_{\phi,\text{loc}}(w) \subset L^1_{\text{loc}}(\mathbf{R}^n)$, where $L_{\phi,\text{loc}}(w)$ denotes the set of functions f such that $\int_Q \phi(|f|)w\,dx < \infty$ for all cubes Q.

(2) Given two weights u and w, with u locally integrable and positive, we can consider the maximal operator $M_{u,w}$ defined by

$$M_{u,w}f(x) = \sup_{x\in Q} \frac{1}{u(Q)} \int_Q |f|\,w\,dx.$$

Then, *if (u, w) satisfies A_ϕ, with u positive, there exists a constant C such that*

$$Mf \leqq C\phi^{-1}(M_{u,w}\phi(|f|)),$$

which generalizes the known inequality $Mf \leqq C(M_{u,w}|f|^p)^{1/p}$, $1 < p < \infty$.

(3) We say that a pair of weights (u, w) satisfies *the doubling condition* if there exists a constant C such that for every cube Q we have that $u(2Q) \leqq Cw(Q)$, where $2Q$ is the double of Q. Taking $f = \chi_Q$ and $2Q$ instead of Q in (2.4) we get that *the A_ϕ-condition implies the doubling condition*.

Now, our problem is to determine those weights w for which there exists a weight $u > 0$ such that M is of weak type (ϕ, ϕ) with respect to (u, w). Using Theorem 2.3 we obtain the following result:

THEOREM 2.7. *Given a weight w on $\mathbf{R}^n$, there exists a weight $u > 0$ such that M is of weak type (ϕ, ϕ) with respect to (u, w) if and only if for every cube I the following two conditions are satisfied*:

(i) *For almost every $x \in \mathbf{R}^n$,*

$$\sup_{\varepsilon>0} \sup_{x\in Q} \varepsilon\varphi\left(\frac{1}{|Q|}\int_Q \rho((\varepsilon w)^{-1}\chi_I)dx\right) < \infty.$$

(ii)
$$\sup_{\varepsilon>0} \sup_{I\subset Q} \varepsilon\varphi\left(\frac{1}{|Q|}\int_Q \rho((\varepsilon|Q|w)^{-1})dx\right) < \infty.$$

PROOF. Assume that there is $u > 0$ such that M is of weak type (ϕ, ϕ) with respect to (u, w) and let I be a cube. Then, w is positive, u is locally integrable and

$$\sup_{\varepsilon>0} \sup_{I\subset Q} \varepsilon\frac{1}{|Q|}\,\varphi\left(\frac{1}{|Q|}\int_Q \rho(1/\varepsilon w)dx\right) \leqq C/u(I) < \infty,$$

where C is an A_ϕ-constant for (u, w), which implies easily (ii).

Since $u \in L^1_{\text{loc}}(\mathbf{R}^n)$ Lebesgue's differentiation theorem asserts that, for almost every $x \in \mathbf{R}^n$,

$$u(x) = \lim_{k\to\infty} \frac{1}{|Q_k|} \int_{Q_k} u\,dx$$

for cubes Q_k containing x and such that $\delta(Q_k) \to 0$. Therefore, for almost every x there exist positive reals $\alpha(x)$ and $\beta(x)$ such that for every $\varepsilon > 0$ and every cube Q, with $|Q| \leqq \alpha(x)$ and $x \in Q$,

$$\varepsilon\varphi\left(|Q|^{-1} \int_Q \rho(\chi_I/\varepsilon w)dx\right) \leqq C/\beta(x). \tag{2.8}$$

Let $Q_1(x)$ be the cube centered at x with $|Q_1(x)| = \alpha(x)$ and let I' be a dilation of I such that $|Q_1(x) \cap I'| > 0$. Then, for every $\varepsilon > 0$ and every cube $Q^*(x)$, centered at x with $|Q^*(x) \cap I| > 0$ and $|Q^*(x)| > \alpha(x)$, we have

$$\varepsilon\varphi\left(|Q^*(x)|^{-1} \int_{Q^*(x)} \rho(\chi_I/\varepsilon w)dx\right) \leqq C|I'|/u(Q_1(x) \cap I').$$

Therefore, there exists a constant $C_1(x) > 0$ such that for every $\varepsilon > 0$ and every cube Q containing x with $|Q| > \alpha(x)$

$$\varepsilon\varphi\left(|Q|^{-1} \int_Q \rho(\chi_I/\varepsilon w)dx\right) \leqq C_1(x). \tag{2.9}$$

Thus, condition (i) follows from (2.8) and (2.9).

Now, suppose that conditions (i) and (ii) are satisfied for every cube I. For a fixed I, let $2I$ be the double of I and u_I the function defined by

$$u_I(x) = \chi_I(x)\left(\sup_{\varepsilon>0} \sup_{x\in Q} \varepsilon\varphi\left(\frac{1}{|Q|}\int_Q \rho(\chi_{2I}/\varepsilon w)dx\right)\right)^{-1}.$$

It follows from (i) that $u_I(x) > 0$ a.e. $x \in I$. It suffices to prove that (u_I, w) satisfies A_ϕ, since decomposing $\mathbf{R}^n$ into a mesh of cubes I_i, whose interiors are disjoint, and taking

$$u(x) = \sum_{i=1}^{\infty} 2^{-i} C_{I_i}^{-1} u_{I_i}(x),$$

where the C_{I_i}'s are A_ϕ-constants for (u_{I_i}, w), we get that (u, w) satisfies A_ϕ and moreover $u(x) > 0$ for a.e. $x \in \mathbf{R}^n$.

We have that there exists $\alpha > 1$ such that $\phi(s) \leqq s\varphi(s) \leqq \alpha\phi(s)$ and $\phi(2s) \leqq \alpha\phi(s)$, $s \geqq 0$ which, with the convexity of ϕ, implies the existence of a constant C such that $\varphi(s + t) \leqq C(\varphi(s) + \varphi(t))$ for every $s, t \geqq 0$. Therefore, given a cube Q and $\varepsilon > 0$ we obtain

$$\left(|Q|^{-1}\int_Q \varepsilon u_I dx\right)\varphi\left(|Q|^{-1}\int_Q \rho(1/\varepsilon w)dx\right)$$
$$\leqq C\left(1+\left(|Q|^{-1}\int_Q \varepsilon u_I dx\right)\varphi\left(|Q|^{-1}\int_{Q-2I}\rho(1/\varepsilon w)dx\right)\right). \tag{2.10}$$

If Q is such that $Q \cap I = \varnothing$ or if $Q \cap I \neq \varnothing$ with $|Q| < 2^{-n}|I|$, then

$$\left(|Q|^{-1}\int_Q \varepsilon u_I dx\right)\varphi\left(|Q|^{-1}\int_Q \rho(1/\varepsilon w)dx\right) \leqq C.$$

Let Q be such that $Q \cap I \neq \varnothing$ and $|Q| \geqq 2^{-n}|I|$. In this situation we have

$$\int_Q u_I dx \leqq |I|\left(\varphi\left(|2I|^{-1}\int_{2I}\rho(1/w)dx\right)\right)^{-1} \tag{2.11}$$

and

$$\varepsilon|Q|^{-1}\varphi\left(|Q|^{-1}\int_{Q-2I}\rho(1/\varepsilon w)dx\right)$$
$$\leqq 2^n\varepsilon C|2Q|^{-1}\varphi\left(|2Q|^{-1}\int_{2Q}\rho(1/\varepsilon w)dx\right), \tag{2.12}$$

where C is a constant that depends only on ϕ.

On the other hand, it follows from (ii) that there is a constant $C(I)$ such that for every $\varepsilon > 0$ and every cube $Q' \supset I$

$$\varepsilon|Q'|^{-1}\varphi\left(|Q'|^{-1}\int_{Q'}\rho(1/\varepsilon w)dx\right) \leqq C(I). \tag{2.13}$$

Since $2Q \supset I$ the A_φ-condition for (u_I, w) follows from (2.10), (2.11), (2.12) and (2.13). Thus, the proof is complete.

In the case $\phi(s) = s^p$, $1 < p < \infty$, condition (i) of Theorem 2.7 reduces to saying that, for every cube I, $M(w^{-1/(p-1)}\chi_I)(x) < \infty$ for a.e. $x \in \mathbf{R}^n$, which is equivalent to $w^{-1/(p-1)}$ being locally integrable, since M is of weak type (1, 1) with respect to Lebesgue-measure. On the other hand, (ii) is equivalent to $\sup_{I\subset Q}|Q|^{-q}\int_Q w^{-q/p}dx < \infty$ holding for every cube I, where q is the conjugate to p. In this way we obtain as a Corollary a result given by J. L. Rubio de Francia in [10]. Precisely

COROLLARY 2.14. *Given p, with $1 < p < \infty$, and a weight w on $\mathbf{R}^n$, there exists a weight $u > 0$ such that M is of weak type (p, p) with respect to (u, w) if and only if $w^{-q/p}$ is locally integrable and for every cube I*

$$\sup_{I\subset Q}|Q|^{-q}\int_Q w^{-q/p}dx < \infty,$$

where q is the conjugate to p and the supremum is taken over cubes Q.

Lastly, in the case $u = w$, we notice that the conclusion of *the extrapolation theorem*, in the theory of A_p-weights, given by J. L. Rubio de Francia in [11] can be easily strengthened to Orlicz spaces. Exactly, Rubio de Francia proves the following:

(2.15) *Let T be a sublinear operator defined on a class of measurable functions in $\mathbf{R}^n$. Let $1 \leqq p^* < \infty$ and $1 < p < \infty$. Suppose that T is bounded in $L^{p^*}(w)$ (respectively of weak type (p^*, p^*) with respect to w) for every weight $w \in A_{p^*}$, with a norm that depends only upon the A_{p^*} constant for w. Then, for every $w \in A_p$, T is bounded in $L^p(w)$ (respectively T is of weak type (p, p) with respect to w), with a norm that depends only upon the A_p-constant for w.* (As a corollary of this result it can be deduced that the boundedness of T in $L^p(w)$ can be obtained from the weak type (p^*, p^*).)

Another proof of the result stated above is given by J. García-Cuerva in [2]. Now, our result is the following:

THEOREM 2.16 (Extrapolation theorem). *Let T be a sublinear operator defined on a class of Lebesgue-measurable functions in $\mathbf{R}^n$. Suppose that for some p^*, with $1 \leqq p^* < \infty$, T satisfies*

$$w\{x \in \mathbf{R}^n : |Tf(x)| > \lambda\} \leqq C\lambda^{-p^*} \int_{\mathbf{R}^n} |f|^{p^*} w \, d\mu \qquad (f \in L^1_{\mathrm{loc}}(\mathbf{R}^n), \lambda > 0)$$

for every weight $w \in A_{p^*}$, where C depends only on the A_{p^*}-constant for w. Then, for every N-function ϕ (which satisfies, together with its complementary N-function, the Δ_2-condition) we have

$$\int_{\mathbf{R}^n} \phi(|Tf|) w \, d\mu \leqq C \int_{\mathbf{R}^n} \phi(|f|) w \, d\mu \qquad (f \in L^1_{\mathrm{loc}}(\mathbf{R}^n))$$

for every $w \in A_\phi$, with the constant C depending only on the A_ϕ-constant for w.

After Theorem 2.16 the results in [1] and [5] (see also IV-3 in [3]) for the maximal Hilbert transform and other singular integral operators extend trivially to Orlicz spaces.

PROOF OF THEOREM 2.16. Take an N-function ϕ and let α_ϕ and β_ϕ be the *upper and lower indices* given by

$$\alpha_\phi = \lim_{s\to 0^+} -\log h_\phi(s)/\log s = \inf_{0<s<1} -\log h_\phi(s)/\log s,$$

$$\beta_\phi = \lim_{s\to\infty} -\log h_\phi(s)/\log s = \sup_{s>1} -\log h_\phi(s)/\log s,$$

where

$$h_\phi(s) = \sup_{t>0}(\phi^{-1}(t)/\phi^{-1}(st)).$$

We have that $0 \leqq \beta_\phi \leqq \alpha_\phi \leqq 1$. On the other hand, $\beta_\phi > 0$ if ϕ satisfies the Δ_2-condition, since given $a > 1$ such that $s\varphi(s) < a\phi(s)$, $s > 0$, the function $s \to s^{-a}\phi(s)$ decreases strictly for $s > 0$, which is equivalent to $\phi^{-1}(st) > s^{1/a}\phi^{-1}(t)$ for every $s > 1$, $t > 0$, and thus $\beta_\phi > a^{-1}$. Likewise, $\alpha_\phi < 1$ if the complementary N-function of ϕ satisfies Δ_2, since in this case there exists $b > 1$ such that $b\phi(s) < s\varphi(s)$, $s > 0$, and, then, $\alpha_\phi \leqq b^{-1}$. We call $q_\phi = \alpha_\phi^{-1} > 1$ and $p_\phi = \beta_\phi^{-1} < \infty$ *the lower and upper exponents of* ϕ, respectively.

In [6] it is proved that *w is in the A_ϕ-class if and only if w is in the A_p-class, where $p = q_\phi$*. Therefore, if $w \in A_\phi$ then $w \in A_r$ for some r such that $1 < r < q_\phi$ (see Theorem IV-2.6 in [3]); r and the A_r-constant for w depend only on the A_ϕ-constant for w. On the other hand, $w \in A_s$ for every $s > r$. Thus, it follows from (2.15) that T is simultaneous of weak type (r, r) and (s, s) with respect to the measure $w\,dx$, where s is such that $p_\phi < s < \infty$, with constants which depend only on the A_ϕ-constant for w. Then, the proof is complete taking into account the following *interpolation theorem*:

THEOREM 2.17. *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{F}, v)$ be two σ-finite measure spaces, ϕ an N-function satisfying, together with its complementary N-function, the Δ_2-condition and q_ϕ, p_ϕ the lower and upper exponents of ϕ. Let $T: L_r + L_s \to \mathfrak{M}(Y)$ be a quasi-additive operator which is simultaneously of weak type (r, r) and (s, s) where $1 \leqq r < q_\phi$, $p_\phi < s \leqq \infty$. Then, T maps $L_\phi(\mu)$ into $L_\phi(v)$ and there exists a constant C, which depends only on the ϕ and the constants for weak types (r, r) and (s, s), such that*

$$\int_Y \phi(|Tf|)dv \leqq C\int_X \phi(|f|)d\mu \qquad (f \in L_\phi(\mu)). \tag{2.18}$$

Theorem 2.17 seems to be included in some general interpolation results (see e.g. [12]). For the sake of completeness we include a simple and direct proof.

PROOF OF THEOREM 2.17. It follows easily from the definition of q_ϕ and the

Δ_2-condition of ϕ that if $0<p<q_\phi$ there exists a constant K such that $\phi(ut) \leqq Ku^p\phi(t)$ for every $t \geqq 0$ and $0 \leqq u \leqq 1$. Likewise, if $p_\phi < q < \infty$ there is a constant A such that $\phi(ut) \leqq Au^q\phi(t)$ for every $t \geqq 0$ and $u \geqq 1$.

Given $f \in L_\phi(\mu)$ and $\lambda > 0$ let $f = f_\lambda + f^\lambda$ where $f_\lambda = f\chi_{B(\lambda)}$ and $B(\lambda) = \{x \in X : |f(x)| > \lambda\}$. If $1 \leqq r < q_\phi$ and $p_\phi < s < \infty$ we have that $f_\lambda \in L_r(\mu)$ and $f^\lambda \in L_s(\mu)$.

By hypothesis we have that there is a constant C such that

$$\nu\{y \in Y : |Tg(y)| > \lambda\} \leqq C\lambda^{-r} \int_X |g|^r d\mu,$$

$$\nu\{y \in Y : |Th(y)| > \lambda\} \leqq C\lambda^{-s} \int_X |h|^s d\mu$$

and $|T(g+h)| \leqq C(|Tg| + |Th|)$ for every $g \in L_r(\mu)$, $h \in L_s(\mu)$ and $\lambda > 0$. Hence

$$\begin{aligned}
\int_Y \phi(|Tf|)d\nu &= \int_0^\infty \varphi(\lambda)\nu\{y \in Y : |Tf(y)| > \lambda\}d\lambda \\
&\leqq \int_0^\infty \varphi(\lambda)\nu\{y \in Y : |Tf_\lambda(y)| > \lambda/2C\}d\lambda \\
&\quad + \int_0^\infty \varphi(\lambda)\nu\{y \in Y : |Tf^\lambda(y)| > \lambda/2C\}d\lambda \\
&\leqq 2^r C^{r+1} \int_{X'} |f(x)|^r \left(\int_0^{|f(x)|} \lambda^{-r}\varphi(\lambda)d\lambda \right) d\mu(x) \\
&\quad + 2^s C^{s+1} \int_{X'} |f(x)|^s \left(\int_{|f(x)|}^\infty \lambda^{-s}\varphi(\lambda)d\lambda \right) d\mu(x),
\end{aligned}$$

where $X' = \{x \in X : |f(x)| > 0\}$.

On the other hand, if $x \in X'$ and p and q are such that $r < p < q_\phi$, $p_\phi < q < s$ we have

$$\begin{aligned}
\int_0^{|f(x)|} \lambda^{-r}\varphi(\lambda)d\lambda &\leqq \alpha K \int_0^{|f(x)|} \lambda^{1-r}(\lambda/|f(x)|)^p \phi(|f(x)|)d\lambda \\
&= \frac{\alpha K}{p-r} |f(x)|^{-r}\phi(|f(x)|)
\end{aligned}$$

and

$$\int_{|f(x)|}^\infty \lambda^{-s}\varphi(\lambda)d\lambda \leqq \frac{\alpha A}{s-q} |f(x)|^{-s}\phi(|f(x)|),$$

where $\alpha > 1$ is such that $s\varphi(s) < \alpha\phi(s)$, $s > 0$. Thus, we obtain (2.18) with constant $\alpha(p-r)^{-1}2^r C^{r+1}K + \alpha(s-q)^{-1}2^s C^{s+1}A$.

The case $s = \infty$ follows trivially from the preceding using the well-known Marcinkiewicz's interpolation theorem or else it may be obtained by a simple direct proof.

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